Universal versus drive-dependent exponents for sandpile models

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We study the scaling relations of the Manna [J. Phys. A **24**, L363 (1992)] model. We found that the avalanche exponent depends crucially on whether one drives the system in the bulk or at the boundary while the cutoff scaling exponent is invariant. Scaling relations relating these exponents are derived for various modes of driving. It is shown numerically that the one dimensional Manna model and a recently introduced ricepile model have the same exponents. Finally, a class of nonconserved self-organized critical models is introduced, and a classification scheme for sandpile models is proposed. [S1063-651X(97)02104-1]

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Following the introduction of the Abelian sandpile model (ASM) of Bak, Tang, and Wiesenfeld (BTW) [1], a number of self-organized critical (SOC) models, which define a number of ''universal classes'' depending on their values of exponents, have been introduced. On the other hand, as for the one dimensional (1D) systems, it seems that sandpile models show not simple finite-size scaling but multifractal scaling behaviors [2].

Recently, however, being inspired by the experiment on ricepiles [3], a model was proposed and demonstrated that shows a simple finite-size scaling in 1D for the avalanche size distribution,

$$P(s)\sim s^{-\tau}f_s(s/L^{\sigma}), \tag{1}$$

with $\tau=1.53$ and $\sigma=2.20$ [4–6]. This ricepile model has a stochastic dynamics in the redistribution process of the slope during an avalanche and is different from most previous models, where the avalanche process is deterministic. It should also be noted that the system is driven only at the top, and the obtained exponent $\tau=1.53$ is large compared with other SOC models, where the exponent is usually less than the mean-field value $\tau=3/2$, a fact that has been explained by [5] by observing that the avalanche dimension D stays invariant, with type of driving, and equals the one obtained from the linear interface model [7].

As for the 2D ASM, it has been shown also that τ can take a larger value when the system is driven at boundary [8,9]; τ has been predicted as

$$\tau=1+\pi/2\alpha, \tag{2}$$

with α being an opening angle at the driving point.

In this paper we study the sandpile model originally introduced by Manna [10], which has stochastic redistribution process as in the ricepile model, for the 1D and 2D systems driven in the bulk and at the boundary. By numerical simulation, we can determine the exponents rather accurately using the scaling relations which hold exactly in the present system. The exponents obtained for the 1D Manna model with a boundary driving are very close to those for the ricepile model, which suggests that the 1D Manna model is in the broad universality class proposed by Paczuski and Boettcher [5]. As for the 2D system, the exponents obtained are close to the ASM, but we found that the fractal dimension for an avalanche is constant for various modes of driving in the Manna model, which cannot be true in the ASM. Based on these observations, we propose classification of the models by the level of stochasticity in the redistribution process.

A version of the Manna model we study here is defined as follows. Consider a lattice in d dimension with open boundaries. At each lattice point, the field variable n_i can take an integer value $n_i=\{0,1,\ldots\}$ counting the number of grains on that site. A grain is added to the n_i of a randomly selected site i iteratively, and an avalanche is initiated when one of the variables n_i exceeds 1. The avalanche propagates by redistributing all the grains on all the sites with $n_i>1$ to their nearest neighbors *randomly and independently* until all the variable n_i's become less than or equal to 1. We employ a parallel updating scheme during the avalanches.

The model differs from the standard sandpile model in having randomness in the local redistribution rules. As already seen in extremum dynamics models [7,11–13], the randomness might give critical behavior also to 1D systems, with which case we begin in the following.

Let us start by introducing some exponents for scaling relations. Following the notation of Ben-Hur and Biham [14], the avalanche size s and its width w scale with the avalanche duration time t as

$$s\sim t^{\gamma_{st}}, \quad w\sim t^{\gamma_{wt}}. \tag{3}$$

This is illustrated in Fig. 1 for our 1D Manna model, in the case where the grain is always added at the boundary. Numerics show that

$$\gamma_{st}=1.48\pm0.03, \quad \gamma_{wt}=0.68\pm0.03. \tag{4}$$

Due to the scaling relations we discuss in the following, these two exponents are enough to characterize the critical behavior of the present models as well as many other sandpile models.

The distributions for the avalanche size s, the duration time t, and the width w are supposed to have the scaling forms

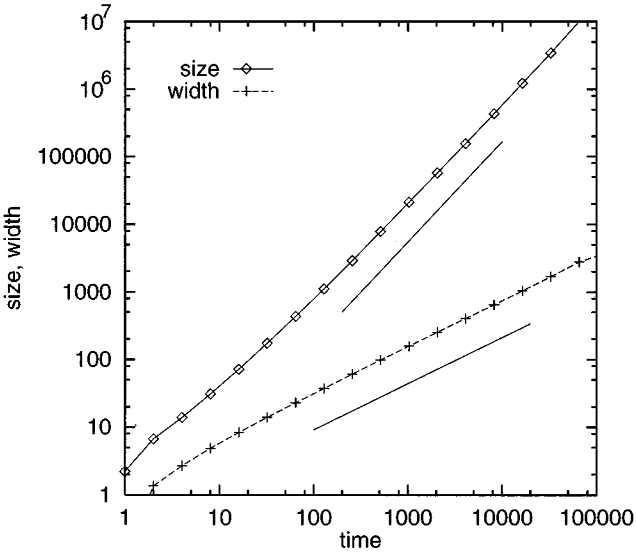

FIG. 1. Simulation results for spreading and the mass of avalanches initiated on the boundary of an open 1D system of size $L=4096$. The lines are to indicate the slopes of 1.48 and 0.68.

$$P(s)=\frac{1}{s^{\tau}}f_s\left(\frac{s}{L^{\sigma}}\right),\quad P(t)=\frac{1}{t^{\tau_t}}f_t\left(\frac{t}{L^{\sigma_t}}\right),$$

$$P(w)=\frac{1}{w^{\tau_w}}f_w\left(\frac{w}{L}\right),\tag{5}$$

where the τ's are the exponents for the distributions and the σ's the exponents for scaling of the cutoff with system size L. These distribution functions are related to each other through the variable transformation (3), thus we can derive the scaling relations

$$\gamma_{st}(\tau-1)=\tau_t-1=\gamma_{wt}(\tau_w-1).\tag{6}$$

Notice that for all sandpile models, the cutoff for the avalanche width must be given directly by the system size L.

As noted by Ben-Hur and Biham [14], there are some obvious relations among these exponents. For example, from Eq. (3) one obtains

$$s\sim w^D,\quad D=\gamma_{st}/\gamma_{wt},\tag{7}$$

where the exponent D is often called the dimension of the avalanche (see the review of Paczuski *et al.* [13]) because it counts how the total mass of the avalanche scales with its spatial extent. The fact that the cutoff for the width w is L gives us the cutoff exponents as

$$\sigma=D=\gamma_{st}/\gamma_{wt},\quad \sigma_t=1/\gamma_{wt}.\tag{8}$$

From the above relations we have reduced the number of independent exponents to three: γ_{st}, γ_{wt}, and τ, for example.

Now we will show that another scaling law can be derived, using the argument which is originally introduced heuristically by Kadanoff *et al.* [2] for sandpile models in general but is *exact* for the present model. We will extend the argument to derive different scaling relations for different ways of driving the system.

If one traces a particular grain, each grain propagates randomly. Each time it topples it does so to left and right with equal probability. Therefore the distance that a particular grain travels is given as an ordinary random walk with time counted by the number of times it has toppled. Notice that this time counting is very different from the real time, where often a particular grain gets stuck for a long time. If one deposits grains randomly in the bulk, the distance that each grain has to travel before it falls out of the system at the boundary is of the order of L, thus the number of topples s_0 each grain goes through is of the order of L^2, which is a contribution of the grain to avalanches while it remains in the system. In the stationary state, every time a new grain is added to the system, one grain should go out of the system on average, therefore the equality

$$\langle s\rangle=\langle s_0\rangle\tag{9}$$

holds, and thus we have $\langle s\rangle\approx L^2$.

This argument was suggested by Kadanoff *et al.* [2] for deterministic versions of the sandpile models, and the equation $\langle s\rangle\approx L^2$ has later been verified analytically for the ASM by Dhar [15]. The implication is that the deterministic updating does not introduce long range correlation between the tumbling directions of the individual grains.

As for the present model, this picture is exact by the definition of the model, and it leads to the scaling law

$$\sigma(2-\tau)=2,\ \text{or}\ D(2-\tau)=2,\tag{10}$$

which should be valid in all dimensions providing that the system is driven in the bulk.

On the other hand, if one deposits grains only at the boundary of the system as in the case of the ricepile model, the average number of steps s_0 that the grain moves before it falls out of the system should be estimated as the number of steps the grain moves before it returns to the original place with the upper cutoff L^2:

$$\langle s_0\rangle\approx\int_0^{L^2}\frac{s_0}{s_0^{3/2}}ds_0\propto L.\tag{11}$$

The upper cutoff represents the case where the grain falls off through the other end of the system. From Eqs. (9) and (11), we obtain the scaling law in this case,

$$\sigma(2-\tau)=1,\ \text{or}\ D(2-\tau)=1,\tag{12}$$

which is valid also in all dimensions when the system is driven at the boundary.

In Fig. 2, we plot $s^{\tau}P(s)$ versus s/L^{σ} with $\sigma=2.20$ and τ given by Eqs. (10) and (12), or $\tau=1.09$ and 1.55, for the bulk and the boundary depositing cases, respectively. As for the bulk deposition case [Fig. 2(a)], the system size dependence in the scaling region persists even in a fairly large system as has been pointed out [14], but the convergence in the cutoff region is quite convincing, thus we can determine

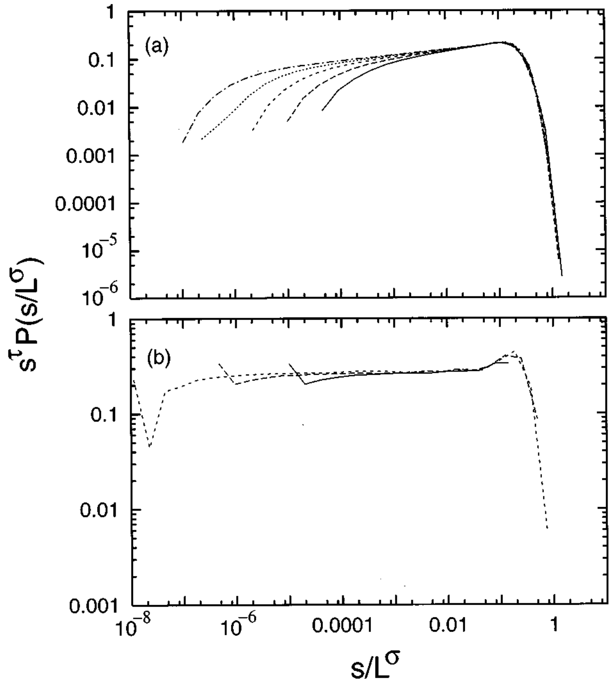

FIG. 2. Finite-size scaling plot for 1D Manna model. We employ $\sigma=2.20$ and $\tau=2-2/\sigma=1.09$ for the bulk driven system (a), and $\sigma=2.20$ and $\tau=2-1/\sigma=1.55$ for the boundary driven system (b). The system sizes are $L=128$, 256, 512, 1024, and 2048 for (a), and $L=256$, 1024, and 4096 for (b).

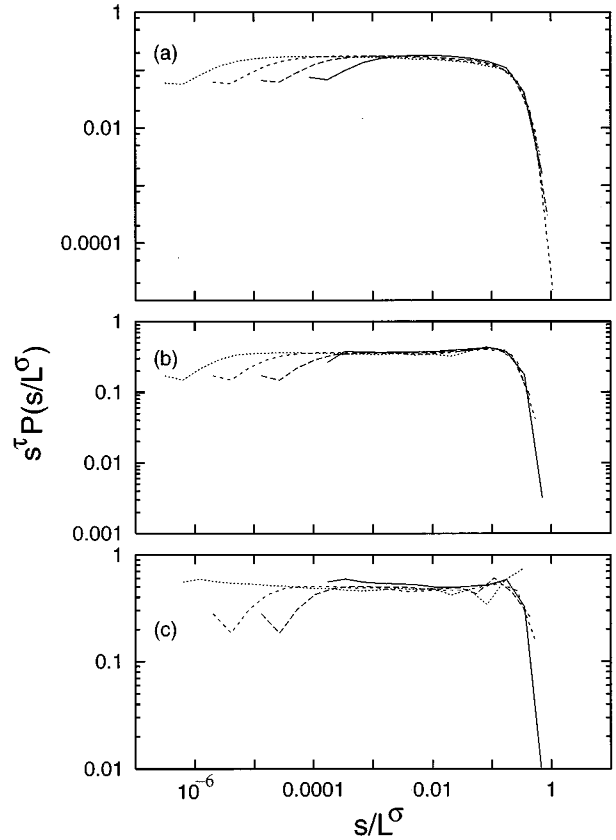

FIG. 3. Finite-size scaling plot for 2D Manna model. We employ $\sigma=2.70$ and $\tau=2-2/\sigma=1.26$ for the bulk driven system (a), $\tau=2-1/\sigma=1.63$ for the boundary driven system (b), and $\tau=2$ for the corner driven system (c). The system sizes are $L=32$, 64, 128, and 256.

the exponent τ through Eq. (10) much better than direct observation of the scaling region. In the case of the boundary deposition case [Fig. 2(b)], overall convergence is very good.

It is noted that Eqs. (10) and (12) are valid also in higher dimensions, and in fact should be valid for all the undirected sandpile models with discrete driving. Furthermore, it is interesting that in higher dimensions one can deposit not only at boundaries, but also at corners of various codimensions. For example, the 2D Manna model can be driven at a corner, then the average number of tumbles s_0 for a grain injected at the corner will be given by the conditional probability that it survives (does not return to 0) during s_0 steps of a random walk both along the x axis and along the y axis. Thus the probability that it survives more than s_0 steps in the lattices is $(1/\sqrt{s_0})1/\sqrt{s_0}=1/s_0$. Therefore the chance that it survives exactly s_0 steps is $1/s_0^2$, thus the mean lifetime for grains in the lattice is

$$\langle s_0\rangle \approx \int_0^{L^2} \frac{s_0}{s_0^2} ds_0 \propto \ln L, \tag{13}$$

implying the scaling relation

$$\sigma(2-\tau)=0, \text{ or } \tau=2 \tag{14}$$

for deposition at a corner in the open boundary 2D lattice. Finite-size scaling results of numerical simulation are given in Fig. 3 for bulk (a), boundary (b), and corner (c) driving with $\sigma=2.70$ and τ's given by Eqs. (10), (12), and (14), respectively: $\tau=1.26$ (a), 1.63 (b), and 2 (c).

Thus, for the two state Manna model, we have reduced in all cases the number of independent exponents to 2, for example, γ_{st} and γ_{wt}, for which we obtained $\gamma_{st}=1.48 \pm 0.03$ and $\gamma_{wt}=0.68\pm 0.03$ in the 1D Manna model. In higher dimensions Ben-Hur and Biham [14] reported that $\gamma_{st}=1.70$ and $\gamma_{wt}=0.67$ for 2D, and $\gamma_{st}=1.80$ and $\gamma_{wt}=0.54$ for 3D, respectively. In all cases the simulated γ's agree well with reported values of τ: 1.09 ± 0.03 for 1D (present work), 1.26 ± 0.03 for 2D (present work).

For both the 1D and 2D systems, the cutoff exponent σ, or the avalanche dimension D, does not depend on the modes of driving the system. Intuitively, this implies that the critical state the system falls in does not depend on the ways of driving, then the stochastic dynamics would extend avalanches in the same way once they go inside of the system. The critical states are examined in Fig. 4, where the densities of zero site n_0 are plotted against $1/L$. It can be seen that n_0 approaches critical density n_c with some power of $1/L$ in the $L\to\infty$ limit, and the critical densities are $n_c=0.107$ for $d=1$ and $n_c=0.319$ for $d=2$, respectively, and they do not depend on the driving.

We will now discuss the connection between the Manna model studied here and other SOC models. First, we should point out that Eq. (12) was derived for the Oslo ricepile model by Ref. [5] using a slightly different picture, and it has

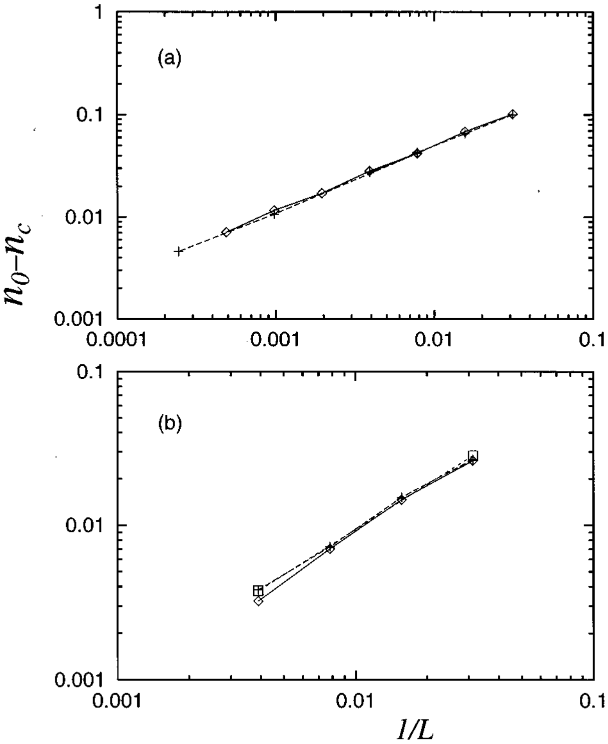

FIG. 4. The system size L dependence of the zero site density n_0. n_0-n_c is plotted against $1/L$ in the logarithmic scale with $n_c=0.107$ for the 1D system (a), and $n_c=0.319$ for the 2D system (b), respectively, for various modes of driving. The data for different driving almost overlap each other.

also been argued that the avalanche dimension D does not depend on the way of driving. In fact, the connection between the Manna model and the ricepile model is very close. The ricepile model may be seen as an integrated version of the Manna model, in the sense that the n_i should be identified as the slopes in the ricepile model. The only difference is then that in the ricepile model the random updates always involve a simultaneous redistribution of n_i to the left and right neighbor, whereas it is random in our formulation of the Manna model. This difference should be insignificant on large scales. In fact, the exponents obtained for the ricepile model ($\sigma=2.20$, $\tau=1.53$) are the same with the 1D Manna model, thus we conclude that the Manna model and the Oslo ricepile model are in the same universality class.

Paczuski and Boettcher [5] have demonstrated that the 1D ricepile model can be mapped to the linear interface model $dH_i/dt=d^2H/dx^2+\eta(x,H)$ when this is driven critically (as done by extremum dynamics) [16]. Thus, as a result, the cutoff exponent σ in the 1D Manna model should indeed equal $1+\chi$, where $\chi=1.25\pm0.05$ [16] is the roughness exponent of the linear interface model in 1D. Further, for the 2D Manna model our obtained value for the exponent $D=2.70$ is very close to the exponent $D=2.725\pm0.020$ for the 2D linear interface model [13]. Thus we conjecture that the stochasticity in the Manna model places it into a fairly large class of models, which include the linear interface model, the Zaitsev model [7], and accordingly also the stochastic version of the Zhang model [17]. As shown by [5], in 1D this class further includes the ricepile model and the train block model of earthquakes [18].

As for the 2D case, the boundary driving exponents for the ASM have been studied [8,9,19,20], and the analytical result for the exponents (2) has been obtained [8]. If one compares the exponents obtained for the ASM from Eq. (2), $\tau=5/4$, 3/2, 2 for $\alpha=2\pi$, π, $\pi/2$, with the present exponents for the 2D Manna model, $\tau=1.26$ (bulk), 1.63 (boundary), 2 (corner), they are quite close to each other. In fact, the equation $\langle s\rangle\approx L^2$ [15], thus Eq. (10) also, holds for the bulk driven ASM too.

However, the similarities stop here. Not only is the Manna model in another universality class than the BTW model, as already demonstrated in [14], but also the dimension of avalanches in 2D for the ASM depends crucially on driving. For the bulk driving, the obtained value $\tau=5/4$ [9] leads to the avalanche dimension of $D=8/3$. On the other hand, $D=2$ for the boundary avalanche in the ASM [15,8] because an avalanche initiated on the boundary cannot topple more than once at each site. Note that the exponents τ for the boundary avalanche in the ASM found in Ref. [8] satisfy Eqs. (12) and (14) when $D=2$. From the present formulas (12) and (14), the exponent τ for the boundary driven ASM can be obtained for higher dimension by assuming $D=d$, which should come from the fact that no sites topple twice in the boundary avalanche in the ASM. For high enough dimension, however, the avalanche loses its compactness and its dimension D becomes less than the embedding dimension d. Presumably this will happen for $d>4$ because $D=4$ is the mean-field value in the ASM [21].

The dependence of D on driving differentiates the ASM qualitatively from the Manna model, where D is independent of driving.

As we have seen the stochasticity plays an important role, it is natural to expect another class is defined by introducing more stochasticity into the model by relaxing the local conservation law of grain. This can be done for the Manna model by adding or removing one grain to and/or from a toppling site before redistribution. The adding and removing are done with equal probability. Then, the local conservation now holds only on average.

This type of modification has also been considered in the ASM [22,23] and changes the system behavior drastically; the mean-field behavior is expected because the system loses correlation [23], and our preliminary simulation for the 1D system shows $\tau=1.50\pm0.05$ (bulk), 1.75 ± 0.05 (boundary) and these results are consistent with the mean-field exponents $\tau=3/2$ (bulk), 7/4 (boundary) [19]. It should be noted that the exponents are also obtained from Eqs. (10) and (12) with the mean-field avalanche dimension $D=4$ [21].

The series of models considered above suggests the classification of models based on degree of randomness in local redistribution rules; Bak-Tang-Wiesenfeld type sandpile model with deterministic avalanche dynamics, the Manna and ricepile type model with stochastic avalanche dynamics but with the strict conservation law, and models with the conservation law which holds only on average.

Note added in proof. After submission of the manuscript, we learned that K. B. Lauritsen succeeded in deriving Eq. (2) for the present model using a similar random walk argument.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[2] L.P. Kadanoff, S.R. Nagel, L. Wu, and S. Zhou, Phys. Rev. A **39**, 6524 (1989).
[3] V. Frette, K. Christensen, A. Malte-Sørensen, J. Feder, T. Jøssang, and P. Meakin, Nature (London) **379**, 49 (1996).
[4] K. Christensen, A. Correl, V. Frette, J. Feder, and T. Jøsang, Phys. Rev. Lett. **77**, 107 (1996).
[5] M. Paczuski and S. Boettcher, Phys. Rev. Lett. **77**, 111 (1996).
[6] L.A.N. Amaral and K.B. Lauritsen, Phys. Rev. E **54**, R4512 (1996).
[7] S.I. Zaitsev, Physica A **189**, 411 (1992).
[8] E.V. Ivashkevich, D.V. Ktitarev, and V.B. Priezzhev, J. Phys. A **27**, L585 (1994).
[9] V.B. Priezzhev, D.V. Ktitarev, and E.V. Ivashkevich, Phys. Rev. Lett. **76**, 2093 (1996).
[10] S.S. Manna, J. Phys. A **24**, L363 (1992).
[11] K. Sneppen, Phys. Rev. Lett. **69**, 3539 (1992).
[12] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[13] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).
[14] A. Ben-Hur and O. Biham, Phys. Rev. E **53**, R1317 (1996).
[15] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[16] H. Leschhorn, Physica **A 195**, 324 (1993).
[17] Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).
[18] M. de Sousa Vieira, Phys. Rev. A **46**, 6288 (1992).
[19] A.L. Stella, C. Tebaldi, and G. Calderelli, Phys. Rev. E **52**, 72 (1995).
[20] G. Calderelli, C. Tebaldi, and A.L. Stella, Phys. Rev. Lett. **76**, 4983 (1996).
[21] C. Tang and P. Bak, J. Stat. Phys. **51**, 797 (1988).
[22] S.S. Manna, L.B. Kiss, and J. Kertész, J. Stat. Phys. **61**, 923 (1990).
[23] H.M. Bröker and P. Grassberger, Europhys. Lett. **30**, 319 (1995).